BOREL COMBINATORICS AND COMPLEXITY

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1. Introduction to Borel Combinatorics

The most straightforward generalizations of finite combinatorial objects often have counter-intuitive behavior: for example, the Banach-Tarski paradox relies on the existence of a perfect matching in the appropriate graph. To eliminate this kind of behavior, one can investigate instead definable (i.e., Borel/measurable/Baire measurable) generalizations of combinatorial objects. This is the main idea behind the field of Borel combinatorics.

A *graph G on a set X*, is a symmetric subset of X^2. In this case $X = V(G)$ is called the vertex set and G is called the edge set. We will call x and y *adjacent/connected/neighbors* if $(x, y) \in G$.

If G is a graph, the *chromatic number of G, $\chi(G)$* is the minimal n, such that G admits an *n-coloring*, that is a map $c : V(G) \to n$ with

$$\forall x, y \in V(G) \ ((x, y) \in G \implies c(x) \neq c(y)).$$

Definition 1.1. Assume that $V(G)$ is a Borel space, we define the *Borel chromatic number of G, $\chi_B(G)$* to be the minimal $n \in \{1, 2, \dots, \aleph_0\}$, such that G admits an *Borel n-coloring*, that is a Borel map $c : V(G) \to n$ with

$$\forall x, y \in V(G) \ ((x, y) \in G \implies c(x) \neq c(y)),$$

here n is endowed with the trivial Polish structure.[1]

If G is a graph, a set $S \subseteq V(G)$ is *G-independent*, if it contains no edges, or formally, if $S^2 \cap G = \emptyset$. It is straightforward to see the following.

Claim 1.2. *$\chi_B(G) \leq n$ iff $V(G)$ can be covered with n-many G-independent Borel sets.*

Recall that a *connected component* of a vertex v of a graph G is the collection of vertices w, such that there is a path from v to w in G, i.e., a sequence of vertices $v_0, \dots, v_n$ with $v_0 = v$, $w = v_n$ and $(v_i, v_{i+1}) \in G$. A *cycle* is an injective sequence of vertices $v_0, \dots, v_n$ with $n > 1$, such that $(v_n, v_0), (v_i, v_{i+1}) \in G$ for all i. A graph is *acylcic* if it contains no cycles. A graph is *d-regular*, if every vertex has exactly d neighbors.

Now let us consider the three examples most important for this lecture.

I. The Basic Example. Let $\alpha \in [0, \pi]$ be such that $\frac{\alpha}{\pi}$ is irrational. Denote by T_α the rotation of the circle, S^1 by α. For $x, y \in S^1$ define

$$xGy \iff T_\alpha(x) = y \vee T_\alpha(y) = x.$$

[1] the reader, not familiar with Borel measurability should take the below claim as a definition.

When we define a graph G as above from a function T_α, we will refer to G as the *symmetrization of* T_α.

Clearly, G is an acyclic 2-regular graph.

Proposition 1.3. $2 = \chi(G) < \chi_B(G) = 3$.

Proof. To show $\chi(G) = 2$ just notice that connected components of G are bi-infinite lines, hence they admit a 2-coloring.

To see $\chi_B(G) \leq 3$ fix some interval I on S^1 with diameter less than α. Clearly I is G-independent. Since α/π is irrational, for every $x \notin I$, there is some $n > 0$ with $T^n(x) \in I$. Then let $c(x) = 2 \iff x \in I$ and $c(x)$ be the parity of the minimal n with $T^n(x) \in I$.

Now, for $\chi_B(G) > 2$ assume that $B_0 \cup B_1 = S^1$ is a Borel 2-coloring. Then, there is an i and a nonempty open interval U with the property that $U \setminus B_i$ is meager. But then (as $2\alpha/\pi$ is also irrational), there is an odd n with $T^n_\alpha(U) \cap U \neq \emptyset$. Now, $T^n_\alpha(U) \cap B_i$ is not meager, as it contains $T^n_\alpha(U) \cap U \cap B_i$. On the other hand $T^n_\alpha(U) \cap B_i$ must be meager, as we started with a coloring and T_α is category preserving. □

The above proof in fact yields that the *Baire-measurable chromatic number* of G is 3, where the Baire measurable chromatic number is defined analogously to the Borel chromatic number, with the coloring required to be only Baire-measurable. Of course, the measure ideal is also rather natural to consider in this context.

Exercise 1.4. *Show that $\chi_\lambda(G) = 3$, where χ_λ is the Lebesgue-measurable chromatic number.*

II. The Shift Graph. Let $[\mathbb{N}]^{\mathbb{N}}$ denote the collection of the infinite subsets of the natural numbers. The *shift-graph*, $\mathcal{G}_\mathcal{S}$ on $[\mathbb{N}]^{\mathbb{N}}$ is defined as the symmetrization of the graph of the *shift-map* $\mathcal{S}$, that is,

$$\mathcal{S}(x) = x \setminus \{\min x\}.$$

Clearly $\mathcal{G}_S$ is acyclic, and *locally finite*, that is, every vertex has finitely many neighbors.

Proposition 1.5. $\chi_B(\mathcal{G}_S) = \aleph_0$.

Proof. The coloring $c(x) = \min x$ shows that $\chi_B(\mathcal{G}_S) \leq \aleph_0$. In fact the following, more general statement is true.

Exercise 1.6. *Assume that G is a locally finite Borel graph. Then $\chi_B(G) \leq \aleph_0$.*

Now, to show that $\chi_B(\mathcal{G}_S)$ is infinite we need the following theorem.

Theorem 1.7 (Galvin-Prikry)**.** *Let $k, l \in \mathbb{N}$ and $c : [\mathbb{N}]^{\mathbb{N}} \to l$ be a Borel coloring. There exists a set $A \in [\mathbb{N}]^{\aleph_0}$ such that $c \restriction [A]^{\mathbb{N}}$ is constant.*

To see our claim, towards contradiction, assume that there is Borel l-coloring c of G_S. Then, by the Galvin-Prikry Theorem there is a set A such that all subsets of A are homogeneous. In particular, $c(A) = c(S(A))$, a contradiction. □

The shift-graph has the following, rather surprising property.

Theorem 1.8 (Kechris-Solecki-Todorčević)**.** *Let $C \subset [\mathbb{N}]^{\mathbb{N}}$ be Borel. Then $\chi_B(\mathcal{G}_S \restriction C) \in \{1, 2, 3, \aleph_0\}$. Moreover, all these chromatic numbers can be realized.*

Proof. Assume that $\mathcal{G}_S \restriction C$ admits a finite Borel coloring $c : V(G) \to k$ with $k \geq 4$. We show that $\mathcal{G}_S \restriction C$ admits a Borel $k-1$-coloring.

Define a new coloring $c'_0(x)$ by $c'_0(x) = c(S(x))$, if $S(x) \in C$ and $c'_0(x) = 0$ otherwise. Note that for any x the color of all preimages of x is the same. Clearly c'_0 is also a Borel k-coloring. Now, define $c'(x)$ by letting $c'(x) = c'_0(x)$ in case this value is $\leq k-2$, and otherwise choose a color not used by the neighbors of x (this is possible, as there are at most two colors used).

Iterating this procedure yields that if $\chi_B(\mathcal{G}_S)$ is finite, then $\chi_B(\mathcal{G}_S) \leq 3$. □

Exercise 1.9. *Show the "moreover" part of the statement.*

III. (The Critical Example, $\mathbb{G}_0$).

Now we define a graph, which turns out to be a fundamental object in descriptive set theory and sits in the core of some of the most important dichotomy theorems. Call a sequence $(s_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{N}^{<\mathbb{N}}$ *appropriate* if for each n we have $|s_n| = n$ and for each $t \in \mathbb{N}^{<\mathbb{N}}$ there is some n with $t \subseteq s_n$.

Definition 1.10. Let $(s_n)_{n\in\mathbb{N}}$ be an appropriate sequence of elements of $\mathbb{N}^{<\mathbb{N}}$. For $x, y \in 2^{\mathbb{N}}$, let $x\mathbb{G}_0 y$ iff

$$x = s_n \frown (i) \frown r \text{ and } y = s_n \frown (1-i) \frown r$$

for some $r \in 2^{\mathbb{N}}$.[2]

Exercise 1.11.
- *$\mathbb{G}_0$ is acyclic,*
- *x and y are on the same connected component of $\mathbb{G}_0$ iff x and y differ in finitely many coordinates.*

Proposition 1.12. $\chi_B(\mathbb{G}_0) > \aleph_0$*, In fact, no non-meager set with the BP is* $\mathbb{G}_0$*-independent.*

Proof. Assume that $c : 2^{\mathbb{N}} \to \aleph_0$ is a Borel coloring. Then, for some i, the set $c^{-1}(i)$ is non-meager. Since $c^{-1}(i)$ is Baire-measurable, there is some neighborhood N_t such that $N_t \setminus c^{-1}(i)$ is meager. In turn, there is some n with $N_{s_n} \setminus c^{-1}(i)$ meager. Since the map $s_n \frown (i) \frown r \mapsto s_n \frown (1-i) \frown r$ is category preserving from N_{s_n} to N_{s_n}, there will be some $x, y \in c^{-1}(i) \cap N_{s_n}$, with $(x, y) \in \mathbb{G}_0$, a contradiction. □

Exercise 1.13. *(Challenge, for a beer) Determine* $\chi_\lambda(\mathbb{G}_0)$.

2. Complexity: The bright side

Our goal now is to understand Borel chromatic numbers of graphs. Intuitively speaking, given a graph, we should decide whether it has a Borel n-coloring or not. The natural way of formalizing this intuition is to consider a family $(G_x)_{x\in 2^{\mathbb{N}}}$ graphs and consider the complexity of the set $\{x : \chi_B(G_x) \leq n\}$. More precisely, a *Borel parametrized family of Borel graphs* is a Borel set $G \subseteq (2^{\mathbb{N}})^3$ so that for each x the set G_x is a Borel graph on $2^{\mathbb{N}}$.

Let $A, B \subseteq 2^{\mathbb{N}}$, we say that $A \leq_W B$ if there is a continuous map $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ with $f^{-1}(B) = A$. If Γ is a collection of subsets of $2^{\mathbb{N}}$, recall that a set $B \subseteq 2^{\mathbb{N}}$ is Γ-*hard*, if for any $A \in \Gamma$, we have $A \leq_W B$. We call a set Γ-*complete* if it is Γ-hard and it is in Γ.

[2]Clearly, this definition depends on the choice of (s_n). Nevertheless, we will abuse the terminology by saying talking about *the* graph $\mathbb{G}_0$. This will not cause any problems as all these graphs are bi-embeddable.

Observe that if Γ is closed under continuous preimages and there is an $A \in \Gamma$ with $2^{\mathbb{N}} \setminus A \notin \Gamma$ and B is Γ-hard then $2^{\mathbb{N}} \setminus B \notin \Gamma$: indeed, otherwise the map witnessing $A \leq_W B$ would also witness $2^{\mathbb{N}} \setminus A \leq_W 2^{\mathbb{N}} \setminus B$, in particular, $2^{\mathbb{N}} \setminus A \in \Gamma$.

Let us consider natural upper and lower bounds on this complexity.

Exercise 2.1. *For each $n \in \{1, 2, \dots, \aleph_0\}$ there exists a Borel parametrized family of Borel graphs such that $\{x : \chi_B(B_x) \leq n\}$ is coanalytic hard.*

Carefully checking the definition of Borel colorings and using further coding arguments one can show the following.

Proposition 2.2. *Let $(G_x)_{x \in 2^{\mathbb{N}}}$ be a Borel parametrized family of Borel graphs. Then $\{x : \chi_B(G_x) \leq n\}$ is Σ^1_2.*

For the sake of this note, let us introduce the following shorthand.

Definition 2.3. Let P be a property of Borel graphs. We say that *deciding P is easy*, if for any Borel parametrized of Borel graphs $(G_x)_{x \in 2^{\mathbb{N}}}$ we have $\{x : P(G_x)\}$ is coanalytic.

We say that *deciding the Borel n-coloring problem is hard* if there is a Borel parametrized of Borel graphs $(G_x)_{x \in 2^{\mathbb{N}}}$ for which $\{x : P(G_x)\}$ is Σ^1_2-complete.

It turns out, that the case of uncountable chromatic numbers can be understood quite well. Recall that a *homomorphism* from a graph G to a graph H is a map from $V(G)$ to $V(H)$, which maps edges to edges.

Theorem 2.4 (Kechris-Solecki-Todorčević [4])**.** *Let G be a Borel graph. Then exactly one of the following holds.*

(1) $\chi_B(G) \leq \aleph_0$
(2) $\mathbb{G}_0 \leq_B G$*, where* $\leq_B$ *stands for Borel homomorphism.*[3]

Before sketching the proof of this statement, let us give another description of the graph $\mathbb{G}_0$.

Assume that $(H_n)_n$ is a sequence of finite graphs and $\phi_n : V(H_{n+1}) \to V(H_n)$ are mappings. Define $\varprojlim H_n$ to a graph on the set

$$\{\bar{x} \in \prod V(H_n) : \forall n \ \phi_n(x_{n+1}) = x_n\},$$

by letting $\bar{x}$ and $\bar{y}$ to be connected if $\exists n_0 \forall n \geq n_0$ we have $(x_n, y_n) \in H_n$.

Now, let $(s_n)_{n \in \mathbb{N}}$ be the sequence from the definition of $\mathbb{G}_0$. Let H_n be the graph on 2^n where x and y are connected iff for some $m < n$ we have $x = s_m \frown i \frown r$ and $y = s_m \frown (1-i) \frown r$. Observe that if ϕ_n is defined by $\phi_n(x) = x \restriction n$, then $\varprojlim H_n$ is the graph $\mathbb{G}_0$. We will think about the sequence $(H_n)_n$ as approximations to $\mathbb{G}_0$.

Proof Sketch (essentially by Miller). First note that if $\chi_B(G) \leq \aleph_0$ and G' admits a Borel homomorphism to G then $\chi_B(G') \leq \aleph_0$. Thus, by Proposition 1.12 the two options are exclusive.

We associate a tree of promises to approximate a homomorphism from $\mathbb{G}_0$ to G: an *approximation* is a map $a_n : V(H_n) \to 2^{<\mathbb{N}}$. We say that a_{n+1} *extends* a_n if for all $x \in H_{n+1}$ we have $a_{n+1}(x) \supsetneq a_n(\phi_n(x))$. We say that an approximation is *reasonable* if there exists a homomorphism $\gamma : H_n \to G$ such that for each $x \in V(H_n)$ we have $\gamma(x) \in N_{a_n(x)}$. In this case, we say that γ *witnesses* a_n.

[3]If $V(G)$ is a Polish space, Borel can be replaced by continuous.

Let T_G be the tree of reasonable approximations. So, now assume that T_G is well-founded. To each leaf of T_G a of length n we associate the set

$$A(a) = \{\gamma(s_n) : \gamma \text{ extends } a\}.$$

The next easy lemma is the key combinatorial insight.

Lemma 2.5. *$A(a)$ is G-independent.*

Proof. Otherwise, there was a reasonable extension a' of a: we could glue together a copy of H_{n+1} using the edge in $A(a)$ and two corresponding copies of H_n. □

Now remove from the space the set $\bigcup_{a \text{ is a leaf}} A(a)$ (this set has a countable coloring). Now the leaves seize to be reasonable approximations, update the tree T_G. Iterate this process countably many times, until T_G vanishes. This yields a countable coloring of G.

Unfortunately, even if T_G is ill-founded, the approximations don't necessarily yield a homomorphism, unless G is closed. However, by starting with a closed set F projecting onto G and modifying the definition of approximations to encompass promises about edges (i.e., basic open nbrhds that intersect F) one can complete this proof.

□

Remark 2.6. Observe that in the above proof $\chi_B(G) \leq \aleph_0$ iff T_G has no infinite branches.

Since well-founded trees form a coanalytic set, some further examination of the complexity of the map $G \mapsto T_G$ yields the following corollary.

Corollary 2.7. *Deciding Borel $\aleph_0$-colorability is easy.*

It turns out that an analogous theorem holds for 2-colorability as well.

Theorem 2.8 (Carroy-Miller-Schrittesser-V)**.** *There exists a Borel graph, $\mathbb{L}_0$ such that for any Borel graph exactly one of the following holds.*

(1) $\chi_B(G) \leq 2$
(2) $\mathbb{L}_0 \leq_B G$*, where* $\leq_B$ *stands for Borel homomorphism.*

The graph $\mathbb{L}_0$ arises from an inverse limit construction analogous to the $\mathbb{G}_0$ case, except that one always adds an odd path instead of a single edge, and keeps the finite graphs paths.

Since the proof of the theorem is based on an idea similar to the proof of $\mathbb{G}_0$, one gets the following.

Corollary 2.9. *Deciding Borel 2-colorability is easy.*

3. Complexity: The dark side

In this section we consider results, which describe the impossibility of understanding Borel chromatic numbers. Our main tool is going to be the shift graph $\mathcal{G}_S$.

Lemma 3.1. *Assume that $C \subset [\mathbb{N}]^{\mathbb{N}}$ is a Borel set and that there exists a Borel $\mathcal{G}_S$-independent set U such that for every $x \in C$ there is an n with $S^n(x) \in U$. Then $\chi_B(G_S \restriction C) \leq 3$.*

Proof. Color every element of U 2. Color elements $x \notin U$ by the parity of the minimal n with $S^n(x) \in U$. $\square$

We will identify elements of $[\mathbb{N}]^{\mathbb{N}}$ with their increasing enumeration, i.e., elements of $\mathbb{N}^{\mathbb{N}}$. Recall that a set $B \subseteq [\mathbb{N}]^{\mathbb{N}}$ is *dominating* if for all $f \in [\mathbb{N}]^{\mathbb{N}}$ there exists a $g \in B$ with $f \leq^* g$ (here $\leq^*$ means that with finitely many exceptions $f(n) \leq g(n)$ holds for all n).

Lemma 3.2 ([2])**.** *Assume that $B \subset [\mathbb{N}]^{\mathbb{N}}$ is a non-dominating Borel set. Then $\chi_B(\mathcal{G}_S \restriction B) \leq 3$.*

Proof. Since B is non-dominating, there exists an f such that for all $g \in B$ there are infinitely many n's with $f(n) > g(n)$. By letting $f'(n) = f(2n)$, we can make sure that for any $g \in B$ there is an n such that $|(rang) \cap [f'(n), f'(n+1))| \geq 2$.

Now, let $x \in U$ iff for the minimal n for which $rang) \cap [f'(n), f'(n+1))$ is nonempty, we have $|(rang) \cap [f'(n), f'(n+1))| = 2$.

We claim that the requirements of Lemma 3.1 are satisfied: indeed, if $g \in U$ then $S(g) \notin U$, since $S(g)$ contains only one element in the corresponding interval determined by f', and, by the choice of f' for each $g \in B$ there is an n with $S^n(g) \in U$. $\square$

Using this lemma one can show the following.

Exercise 3.3. *Show that there is a comeager and measure 1*[4] *Borel set B with $\chi_B(\mathcal{G}_S \restriction B) \leq 3$.*

This observation is important because it is hint of complexity: measure or category cannot detect the large chromatic number of $\mathcal{G}_S$.

Proposition 3.4 ([6],[3])**.** *There exists a Borel set $B \subseteq 2^{\mathbb{N}} \times [\mathbb{N}]^{\mathbb{N}}$ such that $\{x : B_x \text{ is non-dominating}\}$ is analytic complete and if B_x is dominating then $B_x = [\mathbb{N}]^{\mathbb{N}}$.*

Proof. Fix a homeomorphism $2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$, and let T_x denote the tree corresponding to x. Then the set $\{x : T_x \in IF\}$ is analytic complete. Now let $B = \{(x, f) : \forall g \in [T_x]\ (g \not\leq^* f)\}$. Clearly, if T_x is ill-founded then B_x is non-dominating: any $f \in [T_x]$ witnesses this, and if $[T_x] = \emptyset$ then $B_x = [\mathbb{N}]^{\mathbb{N}}$.

We have to verify that B is Borel. We claim that $(x, f) \notin B$ iff there is some n such that for all k there exists an $s_k \in Lev_k(T_x)$ with $s_k(m) \leq f(m)$ for all $m \geq n$. The forward direction is clear, and the backward direction follows from König's lemma. $\square$

Now let B be the Borel set from above and consider the parametrized family of graphs $(\mathcal{G}_S \restriction B_x)_x$. Putting together Lemma 3.2 and Proposition 3.4 we obtain the following corollary.

Corollary 3.5. *$\{x : \chi_B(\mathcal{G}_S \restriction B_x) \leq 3\}$ is analytic complete. In particular, deciding Borel 3-colorability is not easy.*

Using a theorem relying on uniformization one can actually show that in the case of local problems (e.g. colorings) analytic hardness automatically implies Σ^1_2-completeness. This yields the following result.

[4] $[\mathbb{N}]^{\mathbb{N}}$ is identified with a co-countable subset of $2^{\mathbb{N}}$ and let measure mean λ

Theorem 3.6 (Todorčević-V [7]). *Deciding Borel n-colorability is hard for all $2 < n < \aleph_0$.*

It is not hard to check that this theorem rules out any sort of basis result, among other conjectures. One significant downside is that this uses graphs with unbounded degrees (subgraphs of the shift).

Using determinacy methods, Marks has shown the following spectacular result.

Theorem 3.7 (Marks [5]). *There exists a 3-regular acyclic Borel graph G with $\chi_B(G) > 3$.*

Exercise 3.8. *Show that if every degree in a Borel graph is at most d, then it admits a $d+1$-coloring. Hint: use Exercise 1.6 and a greedy algorithm.*

It was suggested that Marks' graph could play the role of a basis for acyclic 3-regular Borel graphs of Borel chromatic number > 3.

The combination of the determinacy method, the Ramsey technique above and, surprisingly, a trick from distributed computing yields the optimal result.

Theorem 3.9 (Brandt-Chang-Grebík-Grunau-Rozhoň-V [1]). *Deciding Borel 3-colorability is hard, even for 3-regular, acyclic Borel graphs.*

One would be tempted to think that the hardness of solving the analogous finitary coloring problems are reflected in the Borel case. Unfortunately, this is false. Similarly to Borel graphs, one can talk about Borel systems of linear equations above some fixed finite field. In the finite world, Gaussian elimination quickly solves such systems. In contrast, we have the following theorem.

Theorem 3.10 (Grebík-V). *Deciding solvability of Borel linear equations over $\mathbb{F}_2$ is hard.*

4. Open Problems

Coloring problems can be reformulated in terms of homomorphism problems. For example, n-coloring is a homomorphism to a complete graph on n-vertices.

Problem 4.1. *Characterize when a homomorphism problem is hard in the Borel context.*

While Theorem 3.9 is optimal in the Borel context, it could be the case that there is still a positive answer for very nice graphs.

Problem 4.2. *Is deciding 3-colorability hard for continuous graphs on compact spaces?*

Also, one could consider slightly more complicated colorings than Borel ones.

Problem 4.3. *Is there a basis for infinitely chromatic subgraphs of the shift, if considers projective colorings?*

A natural question after the 1-2-3-∞ theorem is, that what happens if one allows more functions.

Problem 4.4. *What are the possible Borel chromatic numbers of Borel graphs given by n-many functions?*

Finally, one of the major open problems of the area is to characterize Borel hyperfiniteness.

Problem 4.5. *Is deciding Borel hyperfiniteness hard?*

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